GREEN'S FUNCTION FOR TWO WEIGHTED LAPLACIANS IN THE UNIT DISC

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ABSTRACT. We find Green's function for two weighted Laplacians in the unit disc and express them in terms of the zero-balanced Gauss' hypergeometric function ${}_2F_1(1, \alpha+1; \alpha+2; z)$. The two weights that are considered are $|z|^{2\alpha}$ and $(1-|z|^2)^\alpha$ where $\alpha > -1$. The Bergman kernels for these two weights are calculated using Green's function. The connection to the Poisson kernel is also verified for the known case $(1-|z|^2)^\alpha$ and in the case $|z|^{2\alpha}$ an expression for the kernel is calculated.

CONTENTS

1. INTRODUCTION

In [6] Garabedian showed the existence of Green's function for the following weighted Laplace equation:

$$\frac{\partial}{\partial \bar{z}} \frac{1}{\rho(z)} \frac{\partial}{\partial z} u(z) = 0, \quad z \in \Omega, \tag{1}$$

where Ω is a domain in the complex plane, $\rho : \Omega \to [0, \infty)$ is a positive weight function and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \text{ where } z = x + iy.$$

Physically we can interpret this equation as the complex equivalent of the conductivity equation for the electric potential inside a material with conductivity $\sigma = \frac{1}{\rho}$

$$\nabla \cdot \frac{1}{\rho} \nabla u = 0, \tag{2}$$

where this differential operator is equivalent to the real part of the operator in (1) and our weight ρ is interpreted as the resistivity. This equation can be derived by considering the continuity equation $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$ (here ρ is charge density), which for steady currents reduces to $\nabla \cdot \mathbf{J} = 0$. Then since $\mathbf{J} = \sigma \mathbf{E} = \frac{1}{\rho}\mathbf{E}$ and $\mathbf{E} = \nabla u$, we get $\nabla \cdot \frac{1}{\rho} \nabla u = 0$.

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To see that the operator in (2) is equivalent to the real part of the weighted Laplacian operator in (1) we calculate

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\frac{1}{\rho(z)}\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\\ &= \frac{1}{4}\left(\frac{\partial}{\partial x}\frac{1}{\rho(z)}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\frac{1}{\rho(z)}\frac{\partial}{\partial y}+i\frac{\partial}{\partial y}\frac{1}{\rho(z)}\frac{\partial}{\partial x}-i\frac{\partial}{\partial x}\frac{1}{\rho(z)}\frac{\partial}{\partial y}\right)\end{aligned}$$

and

$$\nabla\cdot\frac{1}{\rho}\nabla = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\cdot\left(\frac{1}{\rho}\frac{\partial}{\partial x},\frac{1}{\rho}\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}\frac{1}{\rho(z)}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\frac{1}{\rho(z)}\frac{\partial}{\partial y}.$$

We conclude that

$$\nabla\cdot\frac{1}{\rho}\nabla = \mathrm{Re}\left[4\frac{\partial}{\partial \bar{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}\right].$$

Among the reasons for studying these weighted problems, the most noteworthy may actually be a kind of inverse of Dirichlet's boundary value problem for the equation (2). This is referred to as Calderon's inverse problem and is stated as finding σ uniquely if we know the so called Dirichlet to Neumann map

$$\Lambda_\sigma : u|_{\partial\Omega} \mapsto \sigma\frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}.$$

Physically we can interpret this map as how different potentials on the surface generates current through the surface, and then Calderon's inverse problem is finding the conductivity inside, which may be used for example in medicine as an imaging tool called Electrical Impedance Tomography. It was solved for conductivities in $W^{1,p}(\Omega)$, $p>2$ and Ω a Lipschitz domain in the plane, by Brown and Uhlmann [4] and later for any bounded simply connected domain Ω in the plane and conductivities in $L^\infty(\Omega)$ by Astala and Päivärinta [2].

Our main object of study will be:

Definition 1. *Green's function* $G_\rho(z,w)$ for the equation (1) in the unit disc $\mathbb{D}$ is defined as a function satisfying, for a fixed w inside $\mathbb{D}$,

i) $-4\frac{\partial}{\partial \bar{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = \delta_0(z-w)$,
ii) $G_\rho(z,w) = 0$ if $z \in \partial\mathbb{D}$.

The coefficient 4 is present to normalize the weighted Laplacian. For if we consider the unweighted case $\rho = 1$ the weighted Laplacian reduces to $\frac{\partial^2}{\partial z\partial \bar{z}} = \frac{1}{4}\Delta$ and so the first condition becomes $-\Delta G_\rho(z,w) = \delta_0$, which is the classical definition (see [5] for the real case).

Note that Green's function has the symmetry property

$$G_\rho(z,w) = \overline{G_\rho(w,z)} \tag{3}$$

so that $\overline{G_\rho(w,z)}$ as a function of w is Green's function for the conjugate of the weighted Laplacian, that is the operator $\frac{\partial}{\partial w}\frac{1}{\rho}\frac{\partial}{\partial \bar{w}}$.

If we know Green's function we can use it to uniquely represent the solution to the Dirichlet boundary value problem

$$\begin{cases} \frac{\partial}{\partial \bar{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}u(z) = 0 & z\in\mathbb{D}\\ u(z) = f(z) & z\in\partial\mathbb{D}.\end{cases} \tag{4}$$

Garabedian considered the case where the boundary value function f is continuously differentiable and in this setting he showed the representation formula for the solution:

$$u(z) = \int_{\partial\mathbb{D}}\frac{f(\zeta)}{\rho(\zeta)}\frac{\partial G_\rho(z,\zeta)}{\partial\nu(\zeta)}ds(\zeta) \tag{5}$$

where s is the arc length and $\nu(\zeta)$ is the inner unit normal at $\zeta \in \partial\mathbb{D}$. Nowadays we usually allow f to be a distribution on $\partial\mathbb{D}$ and consider weak solutions to (4) instead.

The unweighted case $\rho = 1$, often referred to as the classical case, is treated in many books on partial differential equations (for the real case see for instance [5]). Green's function for the unit disc in the classical case is

$$G(z, w) = -\frac{1}{2\pi} \ln|z - w| + \frac{1}{2\pi} \ln|w||z - w^*|$$

where $w^* = \frac{1}{\overline{w}}$ is the reflection (or sometimes called the inversion) of w in the unit circle. This allows us to write

$$G(z, w) = -\frac{1}{2\pi} \ln|z - w| + \frac{1}{2\pi} \ln|1 - z\overline{w}|. \tag{6}$$

Since both our weights are equal to 1 when $\alpha = 0$ it is natural to require that when $\alpha = 0$ our weighted Green's functions should be equal to the classical Green's function.

The aim of this text is to establish Green's function for the weighted Laplacians with weights $|z|^{2\alpha}$ and $(1 - |z|^2)^\alpha$, where $\alpha > -1$. We will also find the Bergman kernels K_ρ associated with the weighted norms

$$||f|| = \int_{\mathbb{D}} |f(z)|^2 \rho(z) dA(z).$$

For information regarding the kernel function we recommend Bergman's book [3]. We will start with the Green's functions and then use the following formula for the Bergman kernel:

$$K_\rho(z, w) = -\frac{4}{\rho(z)\rho(w)} \frac{\partial^2 G_\rho(z, w)}{\partial z \partial \overline{w}}, \quad z \neq w, \tag{7}$$

which was also shown in [6]. Note that this formula differs in the coefficients from the formula found in [6]. This is due to the fact that in Garabedian's version of (5) there is a factor $\frac{1}{2\pi}$ in front of the integral. We have embedded this coefficient in $G_\rho(z, w)$ and therefore the formula for $K_\rho(z, w)$ is adjusted accordingly.

In conclusion, by comparing the representation (5) and the representation using Poisson's integral we will be able to find the Poisson kernel for these two weights. For the weight $\rho(z) = (1 - |z|^2)^\alpha$ the kernel was found in the soon to be published paper [8] by Olofsson and Wittsten. In the other case, $\rho(z) = |z|^{2\alpha}$, we will calculate an expression for it.

The main results of this text are these two Green's functions:

Proposition 1. *Green's function for the weighted Laplacian $\frac{\partial}{\partial \overline{z}} \frac{1}{\rho(z)} \frac{\partial}{\partial z}$ in $\mathbb{D}$ with weight $\rho(z) = |z|^{2\alpha}$, $\alpha > -1$, is given by*

$$G_\rho(z, w) = \frac{1}{4\pi} \left[|z|^{2\alpha} \Psi\left(\frac{z}{w}\right) + |w|^{2\alpha} \Psi\left(\frac{\overline{w}}{\overline{z}}\right) - \Psi\left(\frac{1}{w\overline{z}}\right) - |w|^{2\alpha}|z|^{2\alpha} \Psi(\overline{w}z) \right]$$

where

$$\Psi(z) = z \int_0^1 \frac{t^\alpha}{1 - tz} dt = \frac{z}{\alpha + 1} \, {}_2F_1(1, \alpha + 1; \alpha + 2; z).$$

Note that $\Psi(z)$ has a branch point at $z = 1$, and so we make a branch cut from $z = 1$ to $z = \infty$ along the real axis.

Proposition 2. *Green's function for the weighted Laplacian $\frac{\partial}{\partial \overline{z}} \frac{1}{\rho(z)} \frac{\partial}{\partial z}$ in $\mathbb{D}$ with weight $\rho(z) = (1 - |z|^2)^\alpha$, $\alpha > -1$, is given by*

$$G_\rho(z, w) = \frac{1}{4\pi} \frac{(1 - |w|^2)^\alpha (1 - |z|^2)^\alpha}{(1 - \overline{w}z)^\alpha} \Psi\left(\frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \overline{w}z|^2}\right)$$

where

$$\Psi(z) = z\int_0^1 \frac{t^\alpha}{1-tz}dt = \frac{z}{\alpha+1}\, {}_2F_1(1,\alpha+1;\alpha+2;z), \quad z\in\mathbb{D}.$$

Using these two functions we will be able to show the following corollaries giving the Bergman and Poisson kernels.

Corollary 1. *The Bergman kernel for $\mathbb{D}$ with weight $\rho(z) = |z|^{2\alpha}$, $\alpha > -1$, is given by*

$$K_\rho(z,w) = \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2} + \frac{\alpha}{\pi}\frac{1}{1-z\overline{w}}.$$

Corollary 2. *The Poisson kernel for the weighted Laplace equation (1) in $\mathbb{D}$ with the weight $\rho(z) = |z|^{2\alpha}$, $\alpha > -1$, is given by*

$$P_\rho(z) = \frac{1}{1-\overline{z}} + \frac{z|z|^{2\alpha}}{1-z}.$$

Corollary 3. *The Bergman kernel for $\mathbb{D}$ with weight $\rho(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, is given by*

$$K_\rho(z,w) = \frac{\alpha+1}{\pi}\frac{1}{(1-z\overline{w})^{\alpha+2}}.$$

Corollary 4. *The Poisson kernel for the weighted Laplace equation (1) in $\mathbb{D}$ with the weight $\rho(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, is given by*

$$P_\rho(z) = \frac{(1-|z|^2)^{\alpha+1}}{(1-\overline{z})(1-z)^{\alpha+1}}.$$

2. Prerequisites

In this section we will present the necessary definitions and lemmas regarding the Wirtinger derivatives, the distribution $\frac{\partial}{\partial\overline{z}}\frac{1}{\pi}\frac{1}{z-\zeta}$ and the hypergeometric function ${}_2F_1$. The reader who feels sufficiently informed about such matters may skip to Section 2.4, where we define the auxiliary function $\Psi(z)$.

2.1. **The Wirtinger derivatives.** Let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ be the two Wirtinger differential operators which are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{where } z = x+iy.$$

These operators are also called the Cauchy-Riemann operators. Mainly because if we apply these operators to an analytic function f we get, using the Cauchy-Riemann equations,

$$\frac{\partial f}{\partial z} = f'(z) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = 0.$$

(For details see for example Section 11.1 in [9].) Note also that if f is $\overline{z}$-analytic we get that $\frac{\partial f}{\partial z} = 0$.

With respect to the complex conjugate we get the following behavior:

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \overline{f}}{\partial \overline{z}} \quad \text{and} \quad \overline{\frac{\partial f}{\partial \overline{z}}} = \frac{\partial \overline{f}}{\partial z}. \tag{8}$$

For these operators the usual product rule holds and their chain rules are

$$\frac{\partial(f\circ g)}{\partial z} = \left(\frac{\partial f}{\partial z}\circ g\right)\frac{\partial g}{\partial z} + \left(\frac{\partial f}{\partial \overline{z}}\circ g\right)\frac{\partial \overline{g}}{\partial z},$$
$$\frac{\partial(f\circ g)}{\partial \overline{z}} = \left(\frac{\partial f}{\partial z}\circ g\right)\frac{\partial g}{\partial \overline{z}} + \left(\frac{\partial f}{\partial \overline{z}}\circ g\right)\frac{\partial \overline{g}}{\partial \overline{z}}.$$

Note that if the outer function f is analytical the second term vanishes, since $\frac{\partial f}{\partial \bar{z}} = 0$, and we get the following chain rules instead

$$\begin{aligned} \frac{\partial (f \circ g)}{\partial z} &= \left(\frac{\partial f}{\partial z} \circ g \right) \frac{\partial g}{\partial z}, \\ \frac{\partial (f \circ g)}{\partial \bar{z}} &= \left(\frac{\partial f}{\partial z} \circ g \right) \frac{\partial g}{\partial \bar{z}}. \end{aligned} \tag{9}$$

2.2. **The $\frac{\partial}{\partial \bar{z}}$-lemma.** We will frequently encounter the distribution $\frac{\partial}{\partial \bar{z}} \frac{1}{\pi} \frac{1}{z-\zeta}$ and its complex conjugate. The following lemma states the well-known fact that we can interpret $\frac{1}{\pi} \frac{1}{z-\zeta}$ as the fundamental solution to the $\frac{\partial}{\partial \bar{z}}$-operator.

Lemma 1. *It holds that*

$$\frac{\partial}{\partial \bar{z}} \frac{1}{\pi} \frac{1}{z - \zeta} = \delta_0(z - \zeta)$$

in the sense of distributions.

For a proof see the discussion leading up to the expression (3.1.12) in [7]. But essentially we can consider the action of $\frac{\partial}{\partial \bar{z}} \frac{1}{\pi} \frac{1}{z-\zeta}$ on a test function $\phi(z)$. Using the distributional derivative we see that

$$\frac{\partial}{\partial \bar{z}} \frac{1}{\pi} \frac{1}{z - \zeta} [\phi] = -\frac{1}{\pi} \frac{1}{z - \zeta} \left[\frac{\partial \phi}{\partial \bar{z}} \right] = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \phi(z)}{\partial \bar{z}} \frac{1}{z - \zeta} dA(z).$$

Here we can either use the complex version of Green's theorem or use a simpler, more direct proof as in Lemma 20.3 in [9]. We omit the details. We end up with

$$-\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \phi(z)}{\partial \bar{z}} \frac{1}{z - \zeta} dA(z) = \phi(\zeta)$$

and thus $\frac{\partial}{\partial \bar{z}} \frac{1}{\pi} \frac{1}{z-\zeta} = \delta_0(z - \zeta)$ in the sense of distributions.

2.3. **The hypergeometric function.** The expressions for Green's function will be connected to the following well-known function.

Definition 2. The *hypergeometric function* (sometimes referred to as Gauss' hypergeometric function) can be defined as

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad \text{if } |z| < 1 \tag{10}$$

where $(x)_n$ is Pochhammer's symbol defined using the gamma function as

$$(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} = \begin{cases} 1 & n = 0 \\ x(x + 1) \cdots (x + n - 1) & n = 1, 2, \ldots. \end{cases}$$

Outside of $|z| < 1$ we define ${}_2F_1$ using analytic continuation.

Information and formulas regarding the hypergeometric function can be found in [1], which is only one among many excellent sources.

The hypergeometric functions that we will encounter has a branch point at $z = 1$ and therefore we need to make a branch cut. One usually takes the branch cut extending from $z = 1$ to $z = \infty$ along the real axis.

There exists several integral representations for the hypergeometric function. The one that is of most value to us is called Euler's formula

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - tz)^{-a} dt \tag{11}$$

which is valid when $\text{Re}(c) > 0$ and $\text{Re}(b) > 0$. This formula can be used for the analytic continuation as well as being a useful tool for proving properties. To show this formula note that if $|z| < 1$ we can expand $(1 - tz)^{-a}$ using a binomial series which converges uniformly in $\mathbb{D}$. Then use known properties for the Beta function to simplify the resulting series so that we can identify it with (10). Information and properties about the Beta functions can also be found in [1].

We will only be interested in a special case called the zero-balanced hypergeometric function: ${}_2F_1(1, \alpha + 1; \alpha + 2; z)$, which has the following integral representation if $\alpha > -1$:

$$\begin{aligned} {}_2F_1(1, \alpha + 1; \alpha + 2; z) &= \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha + 1)\Gamma(1)} \int_0^1 t^\alpha (1 - tz)^{-1} dt \\ &= (\alpha + 1) \int_0^1 \frac{t^\alpha}{1 - tz} dt. \end{aligned}$$

2.4. **The auxiliary function $\Psi(z)$.** We define an auxiliary function $\Psi(z)$ which is present in both our Green's functions.

Definition 3. Let $\Psi(z)$ be defined for $\alpha > -1$ as

$$\Psi(z) = z \int_0^1 \frac{t^\alpha}{1 - tz} dt = \frac{z}{\alpha + 1} \, {}_2F_1(1, \alpha + 1; \alpha + 2; z).$$

It has the following properties:

- $\Psi(0) = 0$
- $\Psi(z)$ is analytic inside $\mathbb{D}$ and has an analytic continuation outside of $\mathbb{D}$ except at $z = 1$ where it has a branch point. Therefore make a branch cut from $z = 1$ to $z = \infty$ along the real axis, hence $\Psi(z)$ is analytic on $z \in \mathbb{C} \setminus [1, \infty)$.
- In the unweighted case $\alpha = 0$ it holds that
$$\Psi(z) = z \int_0^1 \frac{1}{1 - tz} dt = [-\ln(1 - tz)]_0^1 = -\ln(1 - z).$$
- Since $\Psi(z)$ is real when z lies in the interval $(-1, 1)$ on the real axis, we can with an application of the identity theorem for analytic functions see that $\overline{\Psi(z)} = \Psi(\overline{z})$ if $z \in \mathbb{C} \setminus [1, \infty)$. To motivate this, set $\Xi(z) = \Psi(z) - \overline{\Psi(\overline{z})}$ and note that it is defined and analytic on $z \in \mathbb{C} \setminus [1, \infty)$. Observe that for $z \in (-1, 1)$ we have $\Xi(z) = 0$. Hence the identity theorem yields $\Xi(z) = 0$ in $z \in \mathbb{C} \setminus [1, \infty)$ and so $\overline{\Psi(z)} = \Psi(\overline{z})$ on the same set.

In our Green's functions we are going to encounter the expression $z^\alpha \Psi(z)$ and we will want to apply the $\frac{\partial}{\partial z}$ operator to it. We proceed by using the product rule and we find that

$$\frac{\partial}{\partial z} (z^\alpha \Psi(z)) = \alpha z^{\alpha - 1} \Psi(z) + z^\alpha \frac{\partial}{\partial z} \Psi(z) = z^\alpha \left[\frac{\alpha}{z} \Psi(z) + \frac{\partial}{\partial z} \Psi(z) \right].$$

So if we define the operator $\mathcal{L}_\alpha = \frac{\alpha}{z} + \frac{\partial}{\partial z}$ and calculate $\mathcal{L}_\alpha \Psi$ we find the derivative by

$$\frac{\partial}{\partial z} (z^\alpha \Psi(z)) = z^\alpha \mathcal{L}_\alpha \Psi.$$

We find the action of $\mathcal{L}_\alpha \Psi$ as follows:

$$\begin{aligned} \mathcal{L}_\alpha \Psi &= \frac{\alpha}{z} \Psi(z) + \frac{\partial}{\partial z} \Psi(z) = \frac{\alpha}{z} z \int_0^1 \frac{t^\alpha}{1 - tz} dt + \frac{\partial}{\partial z} \left[z \int_0^1 \frac{t^\alpha}{1 - tz} dt \right] \\ &= \alpha \int_0^1 \frac{t^\alpha}{1 - tz} dt + \int_0^1 \frac{t^\alpha}{1 - tz} dt + z \frac{\partial}{\partial z} \int_0^1 \frac{t^\alpha}{1 - tz} dt \\ &= (\alpha + 1) \int_0^1 \frac{t^\alpha}{1 - tz} dt + \int_0^1 \frac{z t^{\alpha + 1}}{(1 - tz)^2} dt. \end{aligned}$$

The last integral is calculated using partial integration

$$\int_0^1 \frac{zt^{\alpha+1}}{(1-tz)^2}dt = \left[\frac{t^{\alpha+1}}{1-tz}\right]_0^1 - \int_0^1 \frac{(\alpha+1)t^\alpha}{1-tz}dt = \frac{1}{1-z} - (\alpha+1)\int_0^1 \frac{t^\alpha}{1-tz}dt.$$

Therefore we get some cancellation and arrive at

$$\mathcal{L}_\alpha \Psi = \frac{1}{1-z}$$

which finally gives

$$\frac{\partial}{\partial z}\left(z^\alpha \Psi(z)\right) = \frac{z^\alpha}{1-z}. \tag{12}$$

Remark. Using the integral representation of $\Psi(z)$ we can find a finite expression when $\alpha = n = 1, 2, \ldots$

$$\begin{aligned}
\Psi(z) &= z\int_0^1 \frac{t^n}{1-tz}dt = \{u = tz\} = \frac{z}{z^{n+1}}\int_0^z \frac{u^n}{1-u}du \\
&= \frac{1}{z^n}\int_0^z \frac{u^n}{1-u}du = \frac{1}{z^n}\int_0^z \left[-\sum_{k=0}^{n-1} u^k + \frac{1}{1-u}\right]du \\
&= -\frac{1}{z^n}\sum_{k=1}^{n}\frac{z^k}{k} - \frac{\log(1-z)}{z^n} = -\sum_{k=1}^{n}\frac{1}{kz^{n-k}} - \frac{\log(1-z)}{z^n}.
\end{aligned}$$

3. Green's function for the weight $\rho(z) = |z|^{2\alpha}$

In this section we will first find Green's function for the weight $\rho(z) = |z|^{2\alpha}$ and then proceed to use it to calculate the Bergman kernel.

Proposition 1. *Green's function for the weighted Laplacian $\frac{\partial}{\partial \bar{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}$ in $\mathbb{D}$ with weight $\rho(z) = |z|^{2\alpha}$, $\alpha > -1$, is given by*

$$G_\rho(z,w) = \frac{1}{4\pi}\left[|z|^{2\alpha}\Psi\left(\frac{z}{w}\right) + |w|^{2\alpha}\Psi\left(\frac{\overline{w}}{\overline{z}}\right) - \Psi\left(\frac{1}{w\overline{z}}\right) - |w|^{2\alpha}|z|^{2\alpha}\Psi(\overline{w}z)\right]$$

where

$$\Psi(z) = z\int_0^1 \frac{t^\alpha}{1-tz}dt = \frac{z}{\alpha+1}\, {}_2F_1(1, \alpha+1; \alpha+2; z).$$

The proof will be given in three parts, first we motivate that the claimed expression for Green's function can be made well-defined. We will then proceed to show that it really is Green's function by verifying the definition. The property we consider is that for fixed $w \in \mathbb{D}$ our Green's function must satisfy

$$-4\frac{\partial}{\partial \bar{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = \delta_0(z-w).$$

This will be done in two steps, first applying the $\frac{\partial}{\partial z}$-operator and then applying the $\frac{\partial}{\partial \bar{z}}$-operator. The second property, that $G_\rho(z,w)$ should vanish on $\partial\mathbb{D}$, is not hard to verify.

Part 1 of proof: Well-definedness. This formulation of $G_\rho(z,w)$ is not strictly well-defined when either $z = 0$ or $w = 0$. To see that this is not an issue we consider the limit as $w \to 0$ and see that the limit exists. The other case can be treated similarly.

As w approaches zero the second and the fourth term in G_ρ goes to zero, so what is left to consider is the behavior of the expression

$$|z|^{2\alpha}\Psi\left(\frac{z}{w}\right) - \Psi\left(\frac{1}{w\overline{z}}\right).$$

The limit will depend on whether α equals zero or not. If $\alpha = 0$ this simplifies to

$$-\ln\left(1-\frac{z}{w}\right)+\ln\left(1-\frac{1}{w\overline{z}}\right) = -\ln(w-z)+\ln(\overline{z}w-1)-\ln(\overline{z}) \tag{13}$$

which is well-defined when $w = 0$. If $\alpha \neq 0$ we use the integral representation of $\Psi(z)$ to reformulate

$$\Psi\left(\frac{z}{w}\right) = \frac{z}{w}\int_0^1 \frac{t^\alpha}{1-t\frac{z}{w}}dt = z\int_0^1 \frac{t^\alpha}{w-tz}dt$$

and here the right hand side is well-defined when $w = 0$ and equals $-\frac{1}{\alpha}$. The other term can be treated by a similar application of the integral representation of $\Psi(z)$. □

Part 2 of proof: Applying the $\frac{\partial}{\partial z}$-operator. We treat the first three terms together and then continue to differentiate the fourth term alone. We start by expressing the first three terms by their integral representations

$$\begin{aligned} &|z|^{2\alpha}\Psi\left(\frac{z}{w}\right)+|w|^{2\alpha}\Psi\left(\frac{\overline{w}}{\overline{z}}\right)-\Psi\left(\frac{1}{w\overline{z}}\right) = \\ = \quad &|z|^{2\alpha}\frac{z}{w}\int_0^1\frac{t^\alpha}{1-t\frac{z}{w}}dt+|w|^{2\alpha}\frac{\overline{w}}{\overline{z}}\int_0^1\frac{t^\alpha}{1-t\frac{\overline{w}}{\overline{z}}}dt-\frac{1}{w\overline{z}}\int_0^1\frac{t^\alpha}{1-t\frac{1}{w\overline{z}}}dt \\ = \quad &z\int_0^1\frac{(|z|^2t)^\alpha}{w-tz}dt+\overline{w}\int_0^1\frac{(|w|^2t)^\alpha}{\overline{z}-t\overline{w}}dt-\int_0^1\frac{t^\alpha}{w\overline{z}-t}dt. \end{aligned}$$

With two changes of variable, $s = |z|^2 t$ in the first and $s = |w|^2 t$ in the second, we get that

$$\begin{aligned} &|z|^{2\alpha}\Psi\left(\frac{z}{w}\right)+|w|^{2\alpha}\Psi\left(\frac{\overline{w}}{\overline{z}}\right)-\Psi\left(\frac{1}{w\overline{z}}\right) = \\ = \quad &\int_0^{|z|^2}\frac{t^\alpha}{w\overline{z}-t}dt+\int_0^{|w|^2}\frac{t^\alpha}{w\overline{z}-t}dt-\int_0^1\frac{t^\alpha}{w\overline{z}-t}dt \end{aligned} \tag{14}$$

and we want to apply the $\frac{\partial}{\partial z}$-operator to this expression and interpret the result using Lemma 1. We proceed term by term. The first term can be differentiated according to the Leibniz integral rule as

$$\begin{aligned} \frac{\partial}{\partial z}\int_0^{|z|^2}\frac{t^\alpha}{w\overline{z}-t}dt &= \frac{|z|^{2\alpha}}{w\overline{z}-|z|^2}\overline{z}+\int_0^{|z|^2}\frac{\partial}{\partial z}\frac{t^\alpha}{w\overline{z}-t}dt \\ &= \frac{|z|^{2\alpha}}{w-z}+\pi\int_0^{|z|^2}\frac{t^\alpha}{w}\frac{\partial}{\partial z}\frac{1}{\pi}\frac{1}{\overline{z}-\frac{t}{w}}dt \\ &= \frac{|z|^{2\alpha}}{w-z}+\pi\int_0^{|z|^2}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt. \end{aligned}$$

If the second and third terms are treated similarly we see that

$$\frac{\partial}{\partial z}\int_0^{|w|^2}\frac{t^\alpha}{w\overline{z}-t}dt = \pi\int_0^{|w|^2}\frac{t^\alpha}{w}\frac{\partial}{\partial z}\frac{1}{\pi}\frac{1}{\overline{z}-\frac{t}{w}}dt = \pi\int_0^{|w|^2}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt$$

and

$$\frac{\partial}{\partial z}\int_0^1\frac{t^\alpha}{w\overline{z}-t}dt = \pi\int_0^1\frac{t^\alpha}{w}\frac{\partial}{\partial z}\frac{1}{\pi}\frac{1}{\overline{z}-\frac{t}{w}}dt = \pi\int_0^1\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt.$$

Now let f be defined as a distribution with respect to z in the unit disc by the expression

$$f(z)=\int_0^{|z|^2}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt+\int_0^{|w|^2}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt-\int_0^{1}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt.$$

In view of the above, an application of the $\frac{\partial}{\partial z}$-operator to the expression (14) yields

$$\frac{|z|^{2\alpha}}{w-z}+\pi f(z). \tag{15}$$

We want to show that f is zero in the distributional sense. Therefore take any test function ϕ with support in the unit disc and then we try to calculate the action of f on ϕ. We split f into three terms:

$$f_1(z)=\int_0^{|z|^2}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt,\qquad f_2(z)=\int_0^{|w|^2}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt,$$

$$f_3(z)=\int_0^{1}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt.$$

If we interpret the integral $\int_0^a u_t(z)dt$ of a distribution $u_t(z)$ as

$$\left\langle\int_0^a u_t(z)dt,\phi(z)\right\rangle=\int_0^a\langle u_t(z),\phi(z)\rangle\,dt$$

we get the action of f_2:

$$\begin{aligned}\langle f_2,\phi\rangle&=\left\langle\int_0^{|w|^2}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt,\phi(z)\right\rangle\\&=\int_0^{|w|^2}\frac{t^\alpha}{w}\left\langle\delta_0\left(\overline{z}-\frac{t}{w}\right),\phi(z)\right\rangle dt=\int_0^{|w|^2}\frac{t^\alpha}{w}\phi\left(\frac{t}{\overline{w}}\right)dt.\end{aligned}$$

If we do the same for f_3 we get that

$$\langle f_3,\phi\rangle=\int_0^1\frac{t^\alpha}{w}\phi\left(\frac{t}{\overline{w}}\right)dt.$$

The action of f_1 is a bit more complicated and we begin by making a change of variable:

$$\begin{aligned}f_1(z)&=\int_0^{|z|^2}\frac{t^\alpha}{w}\delta_0\left(\overline{z}-\frac{t}{w}\right)dt=\left\{t=|z|^2s\right\}\\&=\int_0^1\frac{|z|^{2\alpha}s^\alpha}{w}\delta_0\left(\overline{z}-\frac{|z|^2s}{w}\right)|z|^2ds=\int_0^1\frac{|z|^{2(\alpha+1)}s^\alpha}{w}\delta_0\left(\overline{z}-\frac{|z|^2s}{w}\right)ds.\end{aligned}$$

We continue by calculating

$$\begin{aligned}\langle f_1,\phi\rangle&=\left\langle\int_0^1\frac{|z|^{2(\alpha+1)}s^\alpha}{w}\delta_0\left(\overline{z}-\frac{|z|^2s}{w}\right)ds,\phi(z)\right\rangle\\&=\int_0^1\frac{s^\alpha}{w}\left\langle|z|^{2(\alpha+1)}\delta_0\left(\overline{z}-\frac{|z|^2s}{w}\right),\phi(z)\right\rangle ds.\end{aligned}$$

Since $|z|^{2(\alpha+1)}$ is continuous when $\alpha>-1$ we can interpret the action inside the integral above as

$$\left\langle|z|^{2(\alpha+1)}\delta_0\left(\overline{z}-\frac{|z|^2s}{w}\right),\phi(z)\right\rangle=\left\langle\delta_0\left(\overline{z}-\frac{|z|^2s}{w}\right),|z|^{2(\alpha+1)}\phi(z)\right\rangle.$$

For a motivation of this note first that δ_0 is of order zero and then see the remarks after Definition 3.1.1 in [7]. Now let $g(z) = \overline{z} - \frac{|z|^2 s}{w} = \overline{z}\left(1 - \frac{sz}{w}\right)$ so that

$$\delta_0(g(z)) = \frac{\delta_0(z)}{|\det J_g(0)|} + \frac{\delta_0\left(z - \frac{w}{s}\right)}{\left|\det J_g\left(\frac{w}{s}\right)\right|} \tag{16}$$

where $\det J_g(z)$ is the determinant of the Jacobian matrix of g evaluated at z. For a motivation of this see Example 6.13 in [7] together with the observation that the zeros of g are isolated. Explicit calculations give that

$$\begin{aligned} |\det J_g(z)| &= \left|\left|\frac{\partial g}{\partial z}\right|^2 - \left|\frac{\partial g}{\partial \overline{z}}\right|^2\right| = \left|\left|-\frac{s\overline{z}}{w}\right|^2 - \left|1 - \frac{sz}{w}\right|^2\right| \\ &= \left|\frac{s^2|z|^2}{|w|^2} - \left(1 - \frac{sz}{w}\right)\left(1 - \frac{s\overline{z}}{\overline{w}}\right)\right| = \left|2\mathrm{Re}\left[\frac{sz}{w}\right] - 1\right|. \end{aligned}$$

Hence if we evaluate the previous expression at $z = 0$ and $z = \frac{w}{s}$ we get that

$$|\det J_g(0)| = 1 \quad \text{and} \quad \left|\det J_g\left(\frac{w}{s}\right)\right| = 1.$$

Therefore (16) simplifies to

$$\delta_0(g(z)) = \delta_0\left(\overline{z} - \frac{|z|^2 s}{w}\right) = \delta_0(z) + \delta_0\left(z - \frac{w}{s}\right)$$

which we can use together with linearity to see that the action of the first δ_0-function vanish and the resulting action is

$$\left\langle \delta_0\left(\overline{z} - \frac{|z|^2 s}{w}\right), |z|^{2(\alpha+1)}\phi(z)\right\rangle = \frac{|w|^{2(\alpha+1)}}{s^{2(\alpha+1)}}\phi\left(\frac{w}{s}\right).$$

Therefore we can continue with the action of f_1:

$$\begin{aligned} \langle f_1, \phi\rangle &= \int_0^1 \frac{s^\alpha}{w}\frac{|w|^{2(\alpha+1)}}{s^{2(\alpha+1)}}\phi\left(\frac{w}{s}\right) ds = \int_0^1 \frac{1}{w}\frac{|w|^{2(\alpha+1)}}{s^{\alpha+1}}\phi\left(\frac{w}{s}\right)\frac{ds}{s} \\ &= \left\{t = \frac{|w|^2}{s}\right\} = \int_{|w|^2}^{\infty} \frac{t^\alpha}{w}\phi\left(\frac{t}{\overline{w}}\right) dt. \end{aligned}$$

Then since $|w| < 1$ we get that if $t > 1$ we must have $\frac{t}{|w|} > 1$. Hence $\phi\left(\frac{t}{\overline{w}}\right) = 0$ if $t > 1$ since ϕ has support inside the unit disc. Therefore we finally get that

$$\langle f_1, \phi\rangle = \int_{|w|^2}^{1} \frac{t^\alpha}{w}\phi\left(\frac{t}{\overline{w}}\right) dt.$$

Now we have all the parts to evaluate the action of f:

$$\begin{aligned} &\langle f, \phi\rangle = \langle f_1, \phi\rangle + \langle f_2, \phi\rangle - \langle f_3, \phi\rangle \\ &\quad = \int_{|w|^2}^{1} \frac{t^\alpha}{w}\phi\left(\frac{t}{\overline{w}}\right) dt + \int_0^{|w|^2} \frac{t^\alpha}{w}\phi\left(\frac{t}{\overline{w}}\right) dt - \int_0^1 \frac{t^\alpha}{w}\phi\left(\frac{t}{\overline{w}}\right) dt = 0. \end{aligned}$$

We conclude that f is the zero distribution. Therefore we know the result of differentiating the first three terms from (15).

We turn to the fourth term which can be treated using the chain rule (9). To do this set $g(z) = z\overline{w}$ for which $\frac{\partial g}{\partial z} = \overline{w}$. This allows us to write

$$|w|^{2\alpha}|z|^{2\alpha}\Psi(\overline{w}z) = w^\alpha\overline{z}^\alpha g(z)^\alpha\Psi(g(z)).$$

Now if we apply the $\frac{\partial}{\partial z}$-operator we get

$$\frac{\partial}{\partial z}|w|^{2\alpha}|z|^{2\alpha}\Psi(\overline{w}z) = w^\alpha\overline{z}^\alpha\frac{\partial}{\partial z}g(z)^\alpha\Psi(g(z)).$$

Observe here that the outer function $z^\alpha \Psi(z)$ is analytic and can be differentiated using (12). Then by the the chain rule (9)

$$\frac{\partial}{\partial z}|w|^{2\alpha}|z|^{2\alpha}\Psi(\overline{w}z) = w^\alpha \overline{z}^\alpha \frac{f(z)^\alpha}{1-f(z)}\frac{\partial f}{\partial z}$$

which is simplified to

$$\frac{\partial}{\partial z}|w|^{2\alpha}|z|^{2\alpha}\Psi(\overline{w}z) = |w|^{2\alpha}|z|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}$$

Now we have treated all four terms and we have

$$\frac{\partial}{\partial z}G_\rho(z,w) = \frac{1}{4\pi}\left[\frac{|z|^{2\alpha}}{w-z} - |w|^{2\alpha}|z|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}\right]. \tag{17}$$

□

Part 3 of proof: Applying the $\frac{\partial}{\partial \overline{z}}$-operator. We continue towards $\frac{\partial}{\partial \overline{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w)$ by dividing by $\rho(z) = |z|^2$ which yields

$$\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = \frac{1}{4\pi}\left[\frac{1}{w-z} - |w|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}\right]$$

and when we apply the $\frac{\partial}{\partial \overline{z}}$-operator we get

$$\frac{\partial}{\partial \overline{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = \frac{1}{4\pi}\left[\frac{\partial}{\partial \overline{z}}\frac{1}{w-z} - \frac{\partial}{\partial \overline{z}}|w|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}\right].$$

The last term vanish since $\frac{1}{1-z\overline{w}}$ is analytic in $\mathbb{D}$ and therefore must vanish under the $\frac{\partial}{\partial \overline{z}}$-operator. The first term is treated by an application of Lemma 1 and we see that

$$-4\frac{\partial}{\partial \overline{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = \delta_0(z-w).$$

What is left to verify of the properties in the definition of Green's function is the value of $G_\rho(z,w)$ when $z \in \partial\mathbb{D}$. Observe that $z \in \partial\mathbb{D}$ means $|z| = 1$, which implies that $\overline{z} = \frac{1}{z}$ and $\rho(z) = |z|^{2\alpha} = 1$. If we use these facts we end up with

$$G_\rho(z,w) = \frac{1}{4\pi}\left[\Psi\left(\frac{z}{w}\right) + |w|^{2\alpha}\Psi(\overline{w}z) - \Psi\left(\frac{z}{w}\right) - |w|^{2\alpha}\Psi(\overline{w}z)\right] = 0.$$

Thus we have shown that the function $G_\rho(z,w)$ satisfies the definition, and therefore we can conclude that $G_\rho(z,w)$ is our Green's function. □

Remark. Our Green's function has the symmetry property $G_\rho(z,w) = \overline{G_\rho(w,z)}$, essentially since $\Psi(z)$ has the property that $\overline{\Psi(z)} = \Psi(\overline{z})$ if $z \in \mathbb{C} \setminus [1,\infty)$.

Remark. In the unweighted case $\alpha = 0$ we get the classical Green's function. We have already calculated two out of four terms in (13) and the two remaining terms are

$$|w|^{2\alpha}\Psi\left(\frac{\overline{w}}{\overline{z}}\right) \quad \text{and} \quad |w|^{2\alpha}|z|^{2\alpha}\Psi(\overline{w}z)$$

which if $\alpha = 0$ simplify to

$$-\ln\left(1-\frac{\overline{w}}{\overline{z}}\right) = -\ln(\overline{z}-\overline{w}) + \ln(\overline{z}) \quad \text{and} \quad \ln(1-\overline{w}z).$$

If we sum these terms according to the signs in our Green's function and combine this with (13) we see that

$$-\ln(w-z) - \ln(\overline{z}-\overline{w}) + \ln(1-\overline{w}z) + \ln(\overline{z}w-1) = -2\ln|z-w| + 2\ln|1-\overline{w}z|$$

which finally yields the same expression as the classical case (6) when $\alpha = 0$

$$G_\rho(z,w) = -\frac{1}{2\pi}\ln|z-w| + \frac{1}{2\pi}\ln|1-\overline{w}z|.$$

Corollary 1. *The Bergman kernel for $\mathbb{D}$ with weight $\rho(z) = |z|^{2\alpha}$, $\alpha > -1$, is given by*

$$K_\rho(z,w) = \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2} + \frac{\alpha}{\pi}\frac{1}{1-z\overline{w}}.$$

Proof. Using Garabedian's formula (7) we can calculate $K_\rho(z,w)$ as

$$K_\rho(z,w) = -\frac{4}{\rho(z)\rho(w)}\frac{\partial^2 G_\rho(z,w)}{\partial z\partial\overline{w}}, \quad z \neq w.$$

From (17) we get

$$\frac{\partial}{\partial z}G_\rho(z,w) = \frac{1}{4\pi}\left[\frac{|z|^{2\alpha}}{w-z} - |w|^{2\alpha}|z|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}\right] = \frac{|z|^{2\alpha}}{4\pi}\left[\frac{1}{w-z} - |w|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}\right].$$

If we apply the $\frac{\partial}{\partial\overline{w}}$-operator to the above the result is

$$\begin{aligned}\frac{\partial}{\partial\overline{w}}\frac{\partial}{\partial z}G_\rho(z,w) &= \frac{|z|^{2\alpha}}{4\pi}\left[\frac{\partial}{\partial\overline{w}}\frac{1}{w-z} - \frac{\partial}{\partial\overline{w}}|w|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}\right]\\ &= \frac{|z|^{2\alpha}}{4\pi}\left[\pi\delta_0(w-z) - \frac{\partial}{\partial\overline{w}}|w|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}\right].\end{aligned}$$

but since $z \neq w$ we can disregard the δ_0-function:

$$\frac{\partial^2 G_\rho(z,w)}{\partial\overline{w}\partial z} = -\frac{|z|^{2\alpha}}{4\pi}\frac{\partial}{\partial\overline{w}}|w|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}.$$

If we carry out the differentiation and simplify we arrive at

$$\frac{\partial}{\partial\overline{w}}w^\alpha\frac{\overline{w}^{\alpha+1}}{1-z\overline{w}} = |w|^{2\alpha}\left[\frac{\alpha}{1-z\overline{w}} + \frac{1}{(1-z\overline{w})^2}\right]$$

and thus

$$\frac{\partial^2 G_\rho(z,w)}{\partial z\partial\overline{w}} = -\frac{|w|^{2\alpha}|z|^{2\alpha}}{4\pi}\left[\frac{\alpha}{1-z\overline{w}} + \frac{1}{(1-z\overline{w})^2}\right].$$

Therefore we can conclude that

$$K_\rho(z,w) = \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2} + \frac{\alpha}{\pi}\frac{1}{1-z\overline{w}}.$$ □

4. Green's function for the weight $\rho(z) = (1-|z|^2)^\alpha$

We will proceed as in the previous section by first finding Green's function for our second weight $\rho(z) = (1-|z|^2)^\alpha$ and then we use it to calculate the Bergman kernel.

Proposition 2. *Green's function for the weighted Laplacian $\frac{\partial}{\partial\bar{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}$ in $\mathbb{D}$ with weight $\rho(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, is given by*

$$G_\rho(z,w) = \frac{1}{4\pi}\frac{(1-|w|^2)^\alpha(1-|z|^2)^\alpha}{(1-\overline{w}z)^\alpha}\Psi\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2}\right)$$

where

$$\Psi(z) = z\int_0^1 \frac{t^\alpha}{1-tz}dt = \frac{z}{\alpha+1}\,{}_2F_1(1,\alpha+1;\alpha+2;z), \quad z \in \mathbb{D}.$$

Proof. We need to verify that for a fixed $w \in \mathbb{D}$ we have

$$-4\frac{\partial}{\partial\bar{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = \delta_0(z-w).$$

We differentiate G_ρ with respect to z using the chain rule (9). Therefore fix w inside $\mathbb{D}$ and set

$$g(z) = \frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2} = \frac{(1-|w|^2)}{(1-\overline{z}w)}\frac{(1-z\overline{z})}{(1-z\overline{w})}$$

so that we can write

$$G_\rho(z,w) = \frac{1}{4\pi}(1-\overline{z}w)^\alpha g(z)^\alpha \Psi(g(z))\,.$$

With some calculations one can show that $g(z)$ is real and that $0 \le g(z) \le 1$ where $g(z) = 1$ if and only if $z = w$.

The outer function $z^\alpha \Psi(z)$ can be differentiated by (12) and to find the inner derivative we use the product rule

$$\frac{\partial g(z)}{\partial z} = \frac{(1-z\overline{z})}{(1-z\overline{w})}\frac{\partial}{\partial z}\frac{(1-|w|^2)}{(1-\overline{z}w)} + \frac{(1-|w|^2)}{(1-\overline{z}w)}\frac{\partial}{\partial z}\frac{(1-z\overline{z})}{(1-z\overline{w})}.$$

To handle the first term note that $\frac{1}{(1-\overline{z}w)}$ is $\overline{z}$-analytic in $\mathbb{D}$ and therefore vanish under the $\frac{\partial}{\partial z}$-operator. Therefore we only have

$$\frac{\partial g(z)}{\partial z} = \frac{(1-|w|^2)}{(1-\overline{z}w)}\frac{\partial}{\partial z}\frac{(1-z\overline{z})}{(1-z\overline{w})}$$

which is calculated using the product rule as

$$\frac{\partial g}{\partial z} = \frac{1-|w|^2}{1-\overline{z}w}\left[\frac{-\overline{z}}{(1-z\overline{w})} - \frac{(-\overline{w})(1-z\overline{z})}{(1-z\overline{w})^2}\right] = \frac{1-|w|^2}{1-\overline{z}w}\frac{\overline{w}-\overline{z}}{(1-z\overline{w})^2}.$$

Then by the chain rule (9) we get

$$\frac{\partial}{\partial z}G_\rho(z,w) = \frac{1}{4\pi}(1-\overline{z}w)^\alpha \frac{g(z)^\alpha}{1-g(z)}\frac{\partial g}{\partial z}$$

which can be simplified to

$$\frac{\partial}{\partial z}G_\rho(z,w) = \frac{1}{4\pi}\frac{(1-|z|^2)^\alpha(1-|w|^2)^{\alpha+1}}{(w-z)(1-z\overline{w})^{\alpha+1}}. \tag{18}$$

The last derivative is then

$$\frac{\partial}{\partial \overline{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = \frac{\partial}{\partial \overline{z}}\frac{1}{4\pi}\frac{(1-|w|^2)^{\alpha+1}}{(w-z)(1-z\overline{w})^{\alpha+1}} = \frac{1}{4}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\overline{w})^{\alpha+1}}\frac{\partial}{\partial \overline{z}}\frac{1}{\pi}\frac{1}{w-z}$$

where we have used the fact that since $\frac{1}{(1-z\overline{w})^{\alpha+1}}$ is analytic in $\mathbb{D}$ it vanish under the $\frac{\partial}{\partial \overline{z}}$-operator. We continue by applying Lemma 1 to the previous expression and we see that

$$\frac{\partial}{\partial \overline{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = -\frac{1}{4}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\overline{w})^{\alpha+1}}\delta_0(z-w).$$

But this can be simplified by realizing that this function vanish unless $z = w$, hence

$$-4\frac{\partial}{\partial \overline{z}}\frac{1}{\rho(z)}\frac{\partial}{\partial z}G_\rho(z,w) = \delta_0(z-w).$$

Finally it is not difficult to realize that $G_\rho(z,w)$ is zero when $z \in \partial\mathbb{D}$, since then $|z| = 1$ which implies that the expression $(1-|z|^2)$ is zero. Thus we have shown that the function $G_\rho(z,w)$ has the required properties, and therefore we can conclude that $G_\rho(z,w)$ is our Green's function. $\square$

Remark. Note that it has the desired symmetry property $G_\rho(z,w) = \overline{G_\rho(w,z)}$. This is easy to verify once we notice that the argument to the Ψ-function is real and that $\Psi(z)$ is real for $z \in [0,1)$ on the real line.

Remark. In the unweighted case $\alpha = 0$ our Green's function reduces to classical Green's function (6) since for $\alpha = 0$

$$G_\rho(z,w) = -\frac{1}{4\pi}\ln\left(1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}\right)$$

which can be simplified to

$$\begin{aligned} G_\rho(z,w) &= \frac{1}{4\pi}\left(-\ln\left[|1-z\overline{w}|^2 - (1-|z|^2)(1-|w|^2)\right] + \ln|1-z\overline{w}|^2\right) \\ &= -\frac{1}{2\pi}\ln|z-w| + \frac{1}{2\pi}\ln|1-z\overline{w}|. \end{aligned}$$

Corollary 3. *The Bergman kernel for $\mathbb{D}$ with weight $\rho(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, is given by*

$$K_\rho(z,w) = \frac{\alpha+1}{\pi}\frac{1}{(1-z\overline{w})^{\alpha+2}}.$$

Proof. Using Garabedian's formula (7) we can calculate $K_\rho(z,w)$ as

$$K_\rho(z,w) = -\frac{4}{\rho(z)\rho(w)}\frac{\partial^2 G_\rho(z,w)}{\partial z\partial\overline{w}}, \quad z \neq w.$$

But from (18) we already have

$$\frac{\partial G_\rho(z,w)}{\partial z} = \frac{1}{4\pi}\frac{(1-|z|^2)^\alpha(1-|w|^2)^{\alpha+1}}{(w-z)(1-z\overline{w})^{\alpha+1}}.$$

To apply the $\frac{\partial}{\partial\overline{w}}$-operator we begin by using the product rule

$$\begin{aligned} \frac{\partial^2 G_\rho(z,w)}{\partial\overline{w}\partial z} &= \frac{1}{4\pi}\frac{\partial}{\partial\overline{w}}\frac{(1-|z|^2)^\alpha}{(w-z)}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\overline{w})^{\alpha+1}} \\ &= \frac{1}{4\pi}\left[\frac{(1-|w|^2)^{\alpha+1}}{(1-z\overline{w})^{\alpha+1}}\frac{\partial}{\partial\overline{w}}\frac{(1-|z|^2)^\alpha}{(w-z)} + \frac{(1-|z|^2)^\alpha}{(w-z)}\frac{\partial}{\partial\overline{w}}\frac{(1-|w|^2)^{\alpha+1}}{(1-z\overline{w})^{\alpha+1}}\right]. \end{aligned}$$

The first term will vanish after an application of Lemma 1 and after noting that $z \neq w$. With some calculations we can find the second term which becomes

$$\frac{\partial^2 G_\rho(z,w)}{\partial\overline{w}\partial z} = -\frac{\alpha+1}{4\pi}\frac{(1-|z|^2)^\alpha(1-|w|^2)^\alpha}{(1-z\overline{w})^{\alpha+2}}.$$

Therefore we can conclude that

$$K_\rho(z,w) = \frac{\alpha+1}{\pi}\frac{1}{(1-z\overline{w})^{\alpha+2}}.$$ □

5. The Poisson kernel

Now using the two Green's functions from the previous sections we will derive the Poisson kernels for our two weights.

If we know the Poisson kernel, which we will denote as $P_\rho(z)$, we can use the Poisson integral to represent the solution to (4) as

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_\rho(re^{i(\theta-\psi)})f(e^{i\psi})d\psi. \tag{19}$$

If we compare this to the solution formula (5)

$$u(z) = \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\rho(\zeta)}\frac{\partial G_\rho(z,\zeta)}{\partial\nu(\zeta)}ds(\zeta)$$

we see that we can find the Poisson kernel by differentiating Green's function. If we let $z = re^{i\theta}$ and $\zeta = e^{i\psi}$ in the integral (19) we get that

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_\rho(z\overline{\zeta})f(\zeta)d\psi$$

and therefore we can identify

$$P_\rho(z\overline{\zeta}) = \frac{2\pi}{\rho(\zeta)}\frac{\partial G_\rho(z,\zeta)}{\partial\nu(\zeta)}. \tag{20}$$

To find the normal derivative of G_ρ consider the following simple lemma:

Lemma 2. *Let f be defined on some neighborhood around $\partial\mathbb{D}$ and assume that it is differentiable on $\partial\mathbb{D}$. If $f(z) = 0$ for all $z \in \partial\mathbb{D}$ then*

$$\left.\frac{\partial f}{\partial\nu}\right|_{z\in\partial\mathbb{D}} = -\frac{2}{z}\frac{\partial f}{\partial\overline{z}}$$

where $\nu(z)$ is the inward unit normal at z.

Proof. We switch to polar coordinates and see that $f(e^{i\theta}) = 0$ for all θ, which implies that $\left.\frac{\partial f}{\partial\theta}\right|_{r=1} = 0$. If we plug this into the chain rule we get for $r = 1$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{r} = x\frac{\partial f}{\partial r},$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial f}{\partial r}\frac{y}{r} = y\frac{\partial f}{\partial r}.$$

The inward unit normal at $\zeta = x+iy \in \partial\mathbb{D}$ is $\nu(\zeta) = (-x,-y)$ and so the normal derivative is

$$\frac{\partial f}{\partial\nu} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot(-x,-y) = -x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}.$$

Using the previous calculations and the fact that $r^2 = x^2 + y^2 = 1$ we can simplify the normal derivative on $\partial\mathbb{D}$ as

$$\left.\frac{\partial f}{\partial\nu}\right|_{r=1} = -x^2\frac{\partial f}{\partial r} - y^2\frac{\partial f}{\partial r} = -(x^2+y^2)\frac{\partial f}{\partial r} = -\frac{\partial f}{\partial r}.$$

For the Wirtinger derivative $\frac{\partial f}{\partial\overline{z}}$ we can write

$$2\frac{\partial f}{\partial\overline{z}} = 2\,\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = x\frac{\partial f}{\partial r} + iy\frac{\partial f}{\partial r} = z\frac{\partial f}{\partial r}$$

and so we conclude that on $\partial\mathbb{D}$ we have the formula

$$\left.\frac{\partial f}{\partial\nu}\right|_{r=1} = -\frac{2}{z}\frac{\partial f}{\partial\overline{z}}.$$

□

Now we can transform the normal derivative in (20) into something which depends on a derivative we have already calculated. If we use the conjugate property of the Wirtinger derivative (8) and the symmetry property (3) we see that

$$\frac{\partial G_\rho(z,w)}{\partial\overline{w}} = \frac{\partial\overline{G_\rho(w,z)}}{\partial\overline{w}} = \overline{\frac{\partial G_\rho(w,z)}{\partial w}}$$

and this last derivative we have already calculated for both our Green's functions. Therefore we get from the previous lemma and our previous calculation that the identification (20) can be written

$$P_\rho(z\overline{\zeta}) = -\frac{4\pi}{\zeta\rho(\zeta)}\frac{\partial G_\rho(z,\zeta)}{\partial\overline{\zeta}} = -\frac{4\pi}{\zeta\rho(\zeta)}\overline{\frac{\partial G_\rho(\zeta,z)}{\partial\zeta}}. \tag{21}$$

Now we have all we need to find the two Poisson kernels.

Corollary 2. *The Poisson kernel for the weighted Laplace equation (1) in $\mathbb{D}$ with the weight $\rho(z) = |z|^{2\alpha}$, $\alpha > -1$, is given by*

$$P_\rho(z) = \frac{1}{1-\overline{z}} + \frac{z|z|^{2\alpha}}{1-z}.$$

Proof. In (17) we found the first derivative as

$$\frac{\partial G_\rho(z,w)}{\partial z} = \frac{1}{4\pi}\left[\frac{|z|^{2\alpha}}{w-z} - |w|^{2\alpha}|z|^{2\alpha}\frac{\overline{w}}{1-z\overline{w}}\right]$$

which if we rename the variables is equivalent to

$$\frac{\partial G_\rho(\zeta,z)}{\partial \zeta} = \frac{1}{4\pi}\left[\frac{|\zeta|^{2\alpha}}{z-\zeta} - |z|^{2\alpha}|\zeta|^{2\alpha}\frac{\overline{z}}{1-\zeta\overline{z}}\right].$$

Now if we plug this into (21) we find that

$$\begin{aligned} P_\rho(z\overline{\zeta}) &= -\frac{4\pi}{\zeta|\zeta|^{2\alpha}}\overline{\frac{\partial G_\rho(\zeta,z)}{\partial\zeta}} = -\frac{4\pi}{\zeta|\zeta|^{2\alpha}}\frac{1}{4\pi}\left[\frac{|\zeta|^{2\alpha}}{\overline{z}-\overline{\zeta}} - |z|^{2\alpha}|\zeta|^{2\alpha}\frac{z}{1-\overline{\zeta}z}\right] \\ &= \frac{1}{\zeta\overline{\zeta}-\zeta\overline{z}} + |z|^{2\alpha}\frac{\frac{1}{\zeta}z}{1-\overline{\zeta}z}. \end{aligned}$$

Now ζ lies on $\partial\mathbb{D}$ and so $|\zeta|^2 = 1$, $|z|^{2\alpha} = |z\overline{\zeta}|^{2\alpha}$ and $\frac{1}{\zeta} = \overline{\zeta}$. Hence

$$P_\rho(z\overline{\zeta}) = \frac{1}{1-\zeta\overline{z}} + |z\overline{\zeta}|^{2\alpha}\frac{\overline{\zeta}z}{1-\overline{\zeta}z}.$$

and therefore we get the desired expression

$$P_\rho(z) = \frac{1}{1-\overline{z}} + \frac{z|z|^{2\alpha}}{1-z}.$$

□

In the next case we already know the Poisson kernel from [8]. So we want to verify the connection to our Green's function.

Corollary 4. *The Poisson kernel for the weighted Laplace equation (1) in $\mathbb{D}$ with the weight $\rho(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, is given by*

$$P_\rho(z) = \frac{(1-|z|^2)^{\alpha+1}}{(1-\overline{z})(1-z)^{\alpha+1}}.$$

Proof. We proceed exactly as before and in (18) we found that

$$\frac{\partial G_\rho(z,w)}{\partial z} = \frac{1}{4\pi}\frac{(1-|z|^2)^\alpha(1-|w|^2)^{\alpha+1}}{(w-z)(1-z\overline{w})^{\alpha+1}}$$

which if we rename the variables is equivalent to

$$\frac{\partial G_\rho(\zeta,z)}{\partial \zeta} = \frac{1}{4\pi}\frac{(1-|\zeta|^2)^\alpha(1-|z|^2)^{\alpha+1}}{(z-\zeta)(1-\zeta\overline{z})^{\alpha+1}}.$$

If we plug this into (21) we arrive at

$$\begin{aligned} P_\rho(z\overline{\zeta}) &= -\frac{4\pi}{\zeta(1-|\zeta|^2)^\alpha}\overline{\frac{\partial G_\rho(\zeta,z)}{\partial\zeta}} = -\frac{4\pi}{\zeta(1-|\zeta|^2)^\alpha}\frac{1}{4\pi}\frac{(1-|\zeta|^2)^\alpha(1-|z|^2)^{\alpha+1}}{(\overline{z}-\overline{\zeta})(1-\overline{\zeta}z)^{\alpha+1}} \\ &= \frac{(1-|z|^2)^{\alpha+1}}{(\zeta\overline{\zeta}-\zeta\overline{z})(1-\overline{\zeta}z)^{\alpha+1}}. \end{aligned}$$

Now ζ lies on $\partial\mathbb{D}$ and so $|\zeta|^2 = 1$ and $|z|^2 = |z\overline{\zeta}|^2$. This allows us to simplify a little and we get finally

$$P_\rho(z\overline{\zeta}) = \frac{(1-|z\overline{\zeta}|^2)^{\alpha+1}}{(1-\zeta\overline{z})(1-\overline{\zeta}z)^{\alpha+1}}.$$

Therefore we can conclude that the Poisson kernel for the weight $\rho(z) = (1 - |z|^2)^\alpha$ is

$$P_\rho(z) = \frac{(1 - |z|^2)^{\alpha+1}}{(1 - \overline{z})(1 - z)^{\alpha+1}}$$

which is the same expression which Olofsson and Wittsten find in [8]. □

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